Eur. Phys. J. C 22, 757–770 (2002)
Digital Object Identifier (DOI) 10.1007/s100520100829

THE EUROPEAN
PHYSICAL JOURNAL C

Anatomy of one-loop effective action in non-commutative scalar field theories*

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Received: 22 July 2001 / Revised version: 19 October 2001 /
Published online: 7 December 2001 –

Abstract. The one-loop effective action of non-commutative scalar field theory with cubic self-interaction is studied. Utilizing the worldline formulation, both the planar and non-planar parts of the effective action are computed explicitly. We find complete agreement of the result with the Seiberg–Witten limit of a string worldsheet computation and with the standard Feynman diagrammatics. We prove that, in the low-energy and large non-commutativity limit, the non-planar part of the effective action is simplified enormously and is resummable into a quadratic action of scalar open Wilson line operators.

1 Introduction

Non-commutative field theories are field theories defined on non-commutative spacetime, whose coordinates are promoted to operators:

$$\left[x^a, x^b\right] = \mathrm{i}\theta^{ab},$$

and fields are multiplied in terms of the $\star$-product,

$$A(x) \star B(y) := \exp_\star\left(\frac{\mathrm{i}}{2}\partial_x \wedge \partial_y\right) A(x)B(y),$$

implying non-local interactions. Thus, the physical aspects of these theories are suspected to be significantly different from the conventional (commutative) field theory. One of the most significant features is the phenomenon of ultraviolet–infrared (UV–IR) mixing. Motivated partly by this phenomenon, in [1] we have studied the effective action of non-commutative scalar field theories and have found that, remarkably, the non-planar part of the effective action is expressible in terms of *scalar* open Wilson line operators – the scalar counterpart of the open Wilson lines [2–5] in non-commutative gauge theories. Specifically, for $\lambda[\Phi^3]_\star$-theory, the non-planar part of the one-loop effective action is given by

$$\Gamma_{\mathrm{np}}[\Phi] = \frac{\hbar}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} W_k[\Phi]\widetilde{\mathcal{K}_{d/2}}(k \circ k)W_{-k}[\Phi], \quad (1.1)$$

in the low-energy, large non-commutativity limit, where

$$W_k[\Phi] := \mathcal{P}_t \int \mathrm{d}^d x \times \exp_\star\left(-g\int_0^1 \mathrm{d}t|\dot{y}(t)|\Phi(x+y(t))\right) \star \mathrm{e}^{\mathrm{i}k\cdot x}$$

$$(\Phi^n W)_k[\Phi] := \left(-\frac{\partial}{\partial g}\right)^n W_k[\Phi], \quad n = 1,2,3,\cdots, \quad \left(g := \frac{\lambda}{4m}\right) \quad (1.2)$$

denote the *scalar* open Wilson line operators and $\widetilde{\mathcal{K}}$ represents the "propagator" of the state created by the open Wilson lines; see Fig. 1. In [1], much as their counterparts in non-commutative *gauge* theories [2–5], we have also shown that the *scalar* open Wilson line operators are appropriate interpolating operators for "dipoles" – weakly interacting, non-local objects describing excitations in non-commutative field theories. Recall that these dipoles obey the so-called "non-commutative dipole relation":

$$\Delta x^a = \theta^{ab}k_b, \quad (1.3)$$

where k_a and Δx^a denote center-of-mass momentum and dipole moment, respectively. As such, the dipoles are ubiquitous to any non-commutative field theory, an aspect

* Work supported in part for SJR and JTY by BK-21 Initiative in Physics (SNU-Project 2), KOSEF Interdisciplinary Research Grant 98-07-02-07-01-5, and KOSEF Leading Scientist Program, for HTS by KOSEF Brain-Pool Program, and for YK by KRF Grant 2001-015-DP0082
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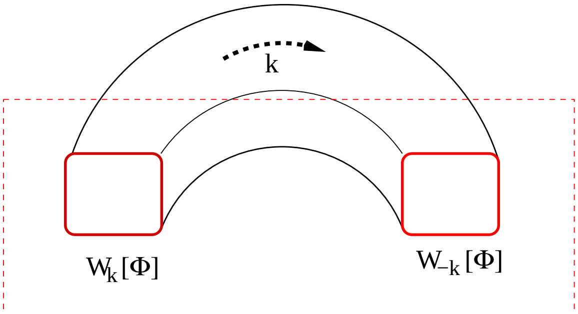

Fig. 1. Scalar open Wilson line representation of the non-planar part of the one-loop effective action. The open Wilson line is an interpolating field of a dipole, whose propagation is governed by the kernel $\mathcal{K}(k \circ k)$ in (1.1)

which would explain why the open Wilson line operators play prominent roles, not only in gauge theories but also in *scalar* field theories, in which neither gauge invariance nor a gauge field is present.

In this paper, in view of the potential importance and far-reaching consequences of the results, (1.1) and (1.2), we present a detailed computation of the one-loop effective action in $\lambda[\Phi^3]_\star$-theory. In [1], the effective action was calculated via the standard Feynman diagrammatics. To supplement the method, in this paper, we will be computing the effective action by the worldline method extended to non-commutative field theories, and make a detailed comparison, wherever possible, with methods and results in the standard Feynman diagrammatics as well as in the point-particle limit of the open string worldsheet computation. For comparison, we compute both the planar and the non-planar parts of the one-loop effective action, but in the low-energy, large non-commutativity limit. In this limit, as is well known, the Weyl–Moyal correspondence enables us to map the non-commutative $\lambda[\Phi^3]_\star$-theory in d dimensions to large-N, $U(N)$ matrix-valued $\lambda\mathrm{Tr}[\Phi^3]$-theory in $(d-2)$ dimensions. The large non-commutativity limit also allows us to recast the non-planar part of the effective action into the form (1.1). Somewhat surprisingly, the planar part of the one-loop effective action is not recastable in terms of closed Wilson loop operators – the putative "master" fields in matrix-valued field theories in the $N \to \infty$ limit.

This paper is organized as follows. In Sect. 2, adopting the worldline formulation, we compute the one-loop N-point, one-particle-irreducible Green functions of non-commutative $\lambda[\Phi^3]_\star$-theory. In doing so, we also observe that non-commutative vertex operators are modified into a form showing the dipole relation (1.3) manifestly. In Sect. 3, we compare the result of Sect. 2 with an open string computation of the N-point S-matrix amplitude at one loop, and find complete agreement in the Seiberg–Witten scaling limit [6]. In Sect. 4, based on the results in Sects. 2 and 3, we compute the one-loop effective action, for both planar and non-planar contributions, by summing over the N-point Green functions. Several remarks and discussions are relegated to the last section.

Our notation is as follows. The spacetime is taken d-dimensional, Wick rotated to Euclidean signature, with metric G_{ab}. Spacetime indices are denoted $a, b, c, \cdots = 1, 2, \cdots, d$. Products involving successively increasing powers of the non-commutativity parameter θ^{ab} are denoted

$$p \cdot q := p_a G^{ab} q_b, \quad p \wedge q := p_a \theta^{ab} q_b,$$
$$p \circ q := p_a (-\theta^2)^{ab} q_b \cdots$$

2 $\lambda[\Phi^3]_\star$-theory: Worldline formulation

Let us begin with the worldline formulation of the non-commutative $\lambda[\Phi^3]_\star$-theory. As stated in the Introduction, we are motivated to do so for a detailed comparison with parallel computatons in the open string theory in the Seiberg–Witten limit. Moreover, the worldline formulation applied to non-commutative field theories, by itself, is of some interest[1].

2.1 The effective action at one-loop

The classical action of the theory is given, after a Wick rotation to Euclidean space, by

$$S_{\mathrm{NC}} = \int \mathrm{d}^d x \left(\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{3!} \Phi \star \Phi \star \Phi \right),$$

or, after a Fourier transform, by

$$S_{\mathrm{NC}} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{2} \widetilde{\Phi}(-k)(k^2 + m^2)\widetilde{\Phi}(k) + \frac{\lambda}{3!} \int \prod_{a=1}^{3} \frac{\mathrm{d}^d k_a}{(2\pi)^d} \widetilde{\Phi}(k_a) \mathrm{e}^{-(\mathrm{i}/2)\sum_{i<j} k_i \wedge k_j} \times (2\pi)^d \delta\left(\sum_{i=1}^{3} k_i\right).$$

The effective action is evaluated most conveniently by utilizing the background field method: split the scalar field $\widetilde{\Phi}$ into $\widetilde{\Phi} = \widetilde{\Phi}_0 + \widetilde{\varphi}$, where $\widetilde{\Phi}_0$ and $\widetilde{\varphi}$ denote classical (background) and quantum (internal) fields, respectively. To one-loop order, only the terms quadratic in $\widetilde{\varphi}$ are relevant. Explicitly, we have

$$S_{\mathrm{NC}} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{\mathrm{d}^d k_2}{(2\pi)^d} \left[(2\pi)^d \delta^d(k_1 + k_2) \frac{1}{2} \left(k_1^2 + m^2\right) + \frac{\lambda}{2} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} (2\pi)^d \delta^d(k_1 + k_2 + k_3) \times \mathrm{e}^{-(\mathrm{i}/2)\sum_{i<j}^{3} k_i \wedge k_j} \widetilde{\Phi}_0(k_3) \right] \widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2) + \cdots \quad (2.1)$$

This shows that, compared to commutative $\lambda[\Phi^3]$-theory, interaction vertices are modified by Moyal's phase factor.

[1] Computations below follow closely the worldline formulation of commutative field theories [7]

These phase factors let the scalar fields be non-commutative while retaining associativity. We can view the scalar fields effectively as matrix-valued fields, whose precise nature is dictated by the so-called Weyl–Moyal correspondence map. Accordingly, the correspondence allows us to classify Feynman diagrams in $\lambda[\Phi^3]_\star$-theory into the planar and the non-planar ones [8–10]. After symmetrization over the momentum labelling, (2.1) is re-expressible in a form suited for dealing with the planar and non-planar diagrams:

$$S_{\rm NC} = \int \frac{{\rm d}^d k_1}{(2\pi)^d}\frac{{\rm d}^d k_2}{(2\pi)^d}(2\pi)^d \Bigg[\frac{1}{2}(k_1^2+m^2)\delta^d(k_1+k_2) + \frac{\lambda}{4}\int\frac{{\rm d}^d p}{(2\pi)^d}\delta^d(k_1+k_2+p) \times \left({\rm e}^{({\rm i}/2)p\wedge k_1}+{\rm e}^{-({\rm i}/2)p\wedge k_1}\right)\widetilde{\Phi}_0(p)\Bigg]\widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2)+\cdots \quad (2.2)$$

Integrating out the quantum fluctuation field $\widetilde{\varphi}$, the one-loop effective action is given schematically by

$$\Gamma_{1-\rm loop}[\Phi_0] = \hbar \ln {\rm Det}^{-1/2}\left[\left(k^2+m^2\right) + \frac{\lambda}{2}\int\frac{{\rm d}^d p}{(2\pi)^d}\left({\rm e}^{({\rm i}/2)p\wedge k}+{\rm e}^{-({\rm i}/2)p\wedge k}\right)\widetilde{\Phi}_0(p)\right]. \quad (2.3)$$

Compared to the one-loop effective action of the commutative $\lambda[\Phi^3]$-theory,

$$\Gamma_{1-\rm loop}[\Phi_0] = \hbar\ln{\rm Det}^{-1/2} \times \left[\left(k^2+m^2\right)+\lambda\int\frac{{\rm d}^d p}{(2\pi)^d}\widetilde{\Phi}_0(p)\right],$$

the interaction vertex is modified by non-commutativity in two ways. First, the coupling parameter is reduced effectively by a factor of 2. Its combinatorial origin is as follows: the entire 3! diagrams can be grouped into two sets of 3 diagrams, related to one another by cyclic permutations. Due to Moyal's phase factors, they constitute inequivalent diagrams. We will refer to the two respective types of interaction vertices in (2.3) as P and T, respectively. Second, the relative sign between P and T terms is *positive*. Recall that, for the *vector* particles as in non-commutative gauge theories, the sign is *negative*. In fact, these signs are attributed to even/odd parity under the worldline time-reversal $\tau \to (1-\tau)$.

For the worldline formulation, we begin with re-expressing the one-loop effective action (2.3) in the path integral representation. In doing so, because of the non-commutativity in P and T, we will need to take care of operator ordering. We thus start with

$$-\ln{\rm Det}\mathcal{F}(k) = \int_0^\infty\frac{{\rm d}T}{T}\int_{x(T)=x(0)}\mathcal{D}x(\tau)\int_{k(T)=k(0)}\mathcal{D}k(\tau) \times\mathcal{P}_\tau\exp\left(-\int_0^T\left[\mathcal{F}(k(\tau))-{\rm i}k(\tau)\cdot\dot{x}(\tau)\right]{\rm d}\tau\right).$$

In theories with non-derivative interactions, such as commutative $\lambda[\Phi^3]$-theory, the function $\mathcal{F}(k)$ is typically quadratic in k, and hence integration over $k(\tau)$ first would be straightforward. In the present case, due to the k-dependent Moyal phase factors in P and T, we proceed differently and expand the background Φ_0 first. The one-loop effective action then comprises terms involving various powers of the P and T, in which $P \to T$ is made by the insertion of "twist" to adjacent internal lines. Explicitly,

$$\begin{aligned}\Gamma_{1-\rm loop}[\Phi_0] &= \frac{\hbar}{2}\int_0^\infty\frac{{\rm d}T}{T}\int\int\mathcal{D}x\mathcal{D}k \times\exp\left(-\int_0^T{\rm d}\tau(k^2+m^2-{\rm i}k\cdot\dot{x})\right)\\ &\times\sum_{N=0}^\infty\sum_{n=0}^N\left(-\frac{\lambda}{2}\right)^N\left[\prod_{\ell=1}^n\int_0^{\tau_{\ell+1}}{\rm d}\tau_\ell\int\frac{{\rm d}^d p_\ell}{(2\pi)^d}\widetilde{\Phi_0}(p_\ell)\right] \times\exp\left(+\frac{\rm i}{2}\sum_{\ell=1}^n p_\ell\wedge k(\tau_\ell)\right)\\ &\times\left[\prod_{j=1}^{N-n}\int_0^{\tau'_{j+1}}{\rm d}\tau'_j\int\frac{{\rm d}^d p'_j}{(2\pi)^d}\widetilde{\Phi_0}(p'_j)\right] \times\exp\left(-\frac{\rm i}{2}\sum_{j=1}^{N-n}p'_j\wedge k(\tau'_j)\right).\end{aligned} \quad (2.4)$$

of n and $(N-n)$ interaction vertices with(out) twists, respectively; see Fig. 2. For each group of vertices, moduli parameters are labeled as τ_ℓ and τ'_j ($\tau_{n+1}=\tau'_{N-n+1}=T$), and external momenta are labeled as p_ℓ and p'_j, respectively. We also assign a sign factor $\nu_l=+1,-1$ to these two groups of interaction vertices. The two square brackets in (2.4) are untwisted and twisted interaction vertices, respectively. Therefore, for given n and N, the one-loop diagram is a function of the following set of momenta and moduli parameters:

$$\begin{aligned}\{\tau_i\} &= \{\tau_{(l)} \quad \text{for} \quad i=1,2,\cdots,n\ ;\\ &\qquad \tau'_{(j)} \quad \text{for} \quad i=n+1,\cdots,N\}\\ \{p_i\} &= \{p_{(l)} \quad \text{for} \quad i=1,2,\cdots,n\ ;\\ &\qquad p'_{(j)} \quad \text{for} \quad i=n+1,\cdots,N\}.\end{aligned}$$

The N-point, one-particle-irreducible (1PI), Green functions are obtained by expanding the effective action (2.4) in powers of Φ_0. In the commutative setup, they are calculated by substituting the classical background field into a sum of "plane waves":

$$\widetilde{\Phi_0}(x(\tau)) \longrightarrow \sum_{\ell=1}^N {\rm e}^{{\rm i}p_\ell\cdot x(\tau)}. \quad (2.5)$$

This substitution is still valid for the present case, as the products between $\widetilde{\Phi_0}$'s in (2.4) in the *momentum* representation are local products (with explicit Moyal's phase

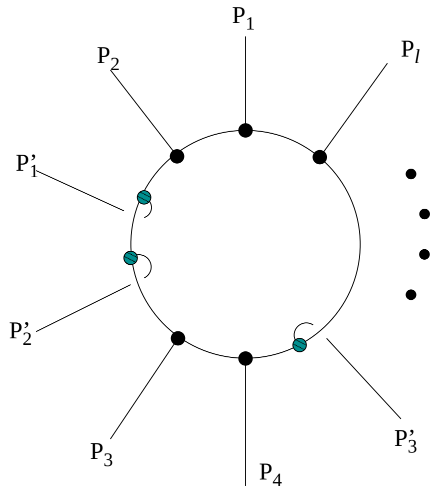

Fig. 2. One-loop N-point Green function. Interaction vertices of untwisted and twisted types are marked with solid and dashed circles, whose momenta are labelled $p_1, \cdots, p_\ell$ and $p'_1, \cdots, p'_{N-n}$

factors attached). By making the plane-wave substitution (2.5), we now generate all the possible diagrams automatically (discarding terms containing the same momentum). This leads, for given n and N, the moduli integrals in (2.4) to result in

$$\left[\prod_{\ell=1}^{n}\int_0^{\tau_{\ell+1}} \mathrm{d}\tau_\ell \widetilde{\Phi_0}\left(p_\ell\right)\right]\cdot\left[\prod_{j=1}^{N-n}\int_0^{\tau'_{j+1}} \mathrm{d}\tau'_j \widetilde{\Phi_0}\left(p'_j\right)\right] \tag{2.6}$$

$$\longrightarrow \prod_{\ell=1}^{n}\int_0^{\tau_{\ell+1}} \mathrm{d}\tau_\ell \prod_{j=1}^{N-n}\int_0^{\tau'_{j+1}} \mathrm{d}\tau'_j$$
$$\times\left[\mathrm{e}^{\mathrm{i}p_1 x(\tau_1)}\mathrm{e}^{\mathrm{i}p_2 x(\tau_2)}\cdots\mathrm{e}^{\mathrm{i}p_N x(\tau_N)} + (\text{all permutations})\right].$$

By interchanging the moduli variables τ, all permutation terms in (2.6) can be arranged so as all possible ordered integrals to have the same integrand. We find that the right-hand side of (2.6) involves the moduli-space integrals:

$$\sum_{\{\nu_i\}}\int_0^T \mathrm{d}\tau_N \cdots \int_0^T \mathrm{d}\tau_1 \prod_{\ell=1}^{N} \mathrm{e}^{\mathrm{i}p_\ell x(\tau_\ell)}$$
$$= \sum_{\{\nu_i\}}\frac{T}{N}\int_0^T \mathrm{d}\tau_{N-1}\cdots\int_0^T \mathrm{d}\tau_1 \prod_{\ell=1}^{N} \mathrm{e}^{\mathrm{i}p_\ell x(\tau_\ell)}.$$

This is essentially the N-point correlator (evaluated with an appropriate worldline Green function). The combinatorics work as follows. The sum over $\{\nu_i\}$ takes into account of all possible 2^N terms, viz. the binomial expansion of $(P+T)^N$ interaction vertices. In the commutative limit, all the 2^N terms reduce to the same contribution, and eventually cancel the $(1/2)^N$ factor originating from the rescaling of the coupling parameter, $\lambda \to \lambda/2$. Alternatively, as the second expression in the above moduli-space integral shows, the sum over $\{\nu_i\}$ takes into account all possible 2^N-terms: $2N$ possibilities for the Nth reference interaction vertex, and 2^{N-1} combinatorial possibilities for the rest. The factor of N is cancelled by the symmetry factor for the reference vertex, $1/N$. The net result is 2^N, yielding the same combinatorial counting.

Consequently, at one loop, the N-point Green function (corresponding to (2.4)) is given by

$$\begin{aligned}&\Gamma_N[p_1,\cdots,p_N]\\&\quad=\frac{\hbar}{2}\left(-\frac{\lambda}{2}\right)^N\sum_{\{\nu_i\}}\int_0^\infty \mathrm{d}T/T\int_0^T\cdots\int_0^T\prod_{\ell=1}^{N}\mathrm{d}\tau_\ell\\&\quad\times\int\mathcal{D}x\int\mathcal{D}k\,\mathrm{e}^{-\int_0^T \mathrm{d}\tau\left[k^2+m^2-\mathrm{i}k\cdot\dot{x}\right]}\\&\quad\times\prod_{j=1}^{N}\mathrm{e}^{\mathrm{i}p_j\cdot x(\tau_j)}\mathrm{e}^{(\mathrm{i}/2)\nu_j p_j\wedge k(\tau_j)}.\end{aligned} \tag{2.7}$$

A brief comment is in order. In the above derivation, for simplicity, we have utilized the plane-wave basis. As will be shown shortly, the phase factor $\exp\left[(\mathrm{i}/2)\nu_j p_j \wedge k(\tau_j)\right]$ ought to be understood as part of a generalized vertex operator, viz. the plane-wave (scalar) vertex operator $\int \mathrm{d}\tau_\ell \mathrm{e}^{\mathrm{i}p_\ell\cdot x(\tau_\ell)}$ is not the proper one in non-commutative field theories. In fact, we will show that the standard Feynman diagrammatics results in Appendix A are reproducible precisely in terms of these new vertex operators.

2.2 The N-point Green function

We next evaluate the path integral in (2.7) explicitly and derive a parametric expression for the one-loop, N-point Green function. In this section, we will prove that the result is given by

$$\begin{aligned}&\Gamma_N[p_1,\cdots,p_N]\\&\quad=\frac{\hbar}{2}\left(-\frac{\lambda}{2}\right)^N\sum_{\{\nu_i\}}\int_0^\infty\frac{\mathrm{d}T}{T}\mathrm{e}^{-m^2T}\left(\frac{1}{4\pi T}\right)^{d/2}\\&\quad\times\int_0^T\prod_{\ell=1}^{N}\mathrm{d}\tau_\ell\prod_{i<j}^{N}\mathrm{e}^{(\mathrm{i}/2)\nu_{ij}\varepsilon(\tau_{ij})p_i\wedge p_j}\\&\quad\times\exp\left[\frac{1}{2}\sum_{k,\ell=1}^{N}p_k\cdot\mathcal{G}_{B\theta}(\tau_k,\tau_\ell;\varepsilon_k,\varepsilon_\ell)\cdot p_\ell\right],\end{aligned} \tag{2.8}$$

where $\mathcal{G}^{ab}_{B\theta}$ denotes the non-commutative counterpart of the worldline propagator $\mathcal{G}_B$:

$$\begin{aligned}&\mathcal{G}^{ab}_{B\theta}\left(\tau_k,\tau_\ell;\varepsilon_k,\varepsilon_\ell\right)=g^{ab}\mathcal{G}_B(\tau_k,\tau_\ell)-\frac{\mathrm{i}}{T}\theta^{ab}\varepsilon_{k\ell}(\tau_k+\tau_\ell)\\&\quad+\frac{1}{4T}(-\theta^2)^{ab}\varepsilon^2_{k\ell}.\end{aligned} \tag{2.9}$$

We have introduced the following notation:

$$\nu_{ij} := \frac{\nu_i+\nu_j}{2},\quad \tau_{ij} := \tau_i-\tau_j,\quad \varepsilon_i := \frac{1-\nu_i}{2},$$

$$\text{and} \quad \varepsilon(\tau) := \text{sign}(\tau).$$

In addition, $\varepsilon_{k\ell}$ refers to $\varepsilon_{k\ell} = \varepsilon_k - \varepsilon_\ell$. In the present case, as (2.4) indicates, the untwisted and the twisted vertices ought to be distinguished from each other, as the exponent of the $\star$-product flips sign, depending on whether the vertex is twisted or not (cf. (2.3)). One recognizes also that the $\star$-product structure is dressed with a twist-dependent "weight":

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}p_i x(\tau_i)} *^{\nu} \mathrm{e}^{\mathrm{i}p_j x(\tau_j)} \\ &\quad = \exp\left(\frac{\mathrm{i}}{2}\nu_{ij}\varepsilon(\tau_{ij}) p_i \wedge p_j\right) \mathrm{e}^{\mathrm{i}p_i x(\tau_i) + \mathrm{i}p_j x(\tau_j)}, \end{aligned}$$

where $\nu_{ij} = 0, +1, -1$ depending on the types of boundaries along which the interaction vertices are inserted.

In the rest of this subsection, we prove (2.8) and (2.9). Start with the path integral over $k(\tau)$. The relevant integral is

$$\begin{aligned} K := \int \mathcal{D}k \exp\Bigg(- \int_0^T \mathrm{d}\tau \Bigg[& k^2(\tau) - \mathrm{i}k \cdot \dot{x}(\tau) \\ & + \frac{\mathrm{i}}{2}\sum_{\ell=1}^{N} \nu_\ell \delta(\tau - \tau_\ell) k(\tau) \wedge p_\ell \Bigg]\Bigg), \end{aligned}$$

a Gaussian type integral. After shifting the momentum density by

$$k^a(\tau) \longrightarrow k^a(\tau) + \frac{\mathrm{i}}{2}\dot{x}^a(\tau) - \frac{\mathrm{i}}{4}\sum_{\ell=1}^{N} \delta(\tau - \tau_\ell)\nu_\ell \theta^{ab} p_\ell^b,$$

the integral yields

$$K = \mathcal{N}(T) \exp\left(-\frac{1}{4}\int_0^T \mathrm{d}\tau \left[\dot{x} - \frac{1}{2}\sum_{\ell=1}^{N} \theta \cdot p_\ell \nu_\ell \delta(\tau - \tau_\ell) \right]^2 \right),$$

where $\mathcal{N}(T)$ is a T-dependent normalization factor, which turns out to be the same as in the commutative case. The square of the δ-functions in the exponent does not lead to divergences, as all τ_ℓ take different values because of the non-commutativity. Thus, one finally finds

$$K = \mathcal{N}(T) \exp\left(-\frac{1}{4}\int_0^T \dot{x}^2 \mathrm{d}\tau \right) \prod_{\ell=1}^{N} \exp\left(\frac{1}{4}\nu_\ell \dot{x}(\tau_\ell) \wedge p_\ell \right).$$

Utilizing the expression for K, the N-point correlation function is then reduced to

$$\begin{aligned} &\Gamma_N[p_1, \cdots, p_N] \\ &= \frac{1}{2}\left(-\frac{\lambda}{2}\right)^N \sum_{\{\nu_i\}} \int_0^\infty \frac{\mathrm{d}T}{T} \mathrm{e}^{-m^2 T} \int_0^T \prod_{\ell=1}^{N} \mathrm{d}\tau_\ell \\ &\quad \times \mathcal{N}(T) \int_{x(0)=x(T)} \mathcal{D}x \exp\left(-\frac{1}{4}\int_0^T \dot{x}^2 \mathrm{d}\tau \right) \\ &\quad \times \prod_{\ell=1}^{N} \mathrm{e}^{\mathrm{i}p_\ell x(\tau_\ell)} \mathrm{e}^{(1/4)\dot{x}(\tau_\ell) \wedge p_\ell \nu_\ell}. \end{aligned} \tag{2.10}$$

Next, evaluate the path integral over $x(\tau)$:

$$\begin{aligned} X := & \int_{x(0)=x(T)} \mathcal{D}x \mathrm{e}^{-(1/4)\int_0^T \dot{x}^2 d\tau} \\ & \times \prod_{\ell=1}^{N} \exp\left(\mathrm{i}p_\ell \cdot x(\tau_\ell) - \frac{\nu_\ell}{4} p_\ell \wedge \dot{x}(\tau_\ell) \right). \end{aligned}$$

The integrand suggests that the vertex operator relevant for the non-commutative scalar field is not the conventional plane-wave vertex operator but, as mentioned earlier,

$$V_{\mathrm{NC}}(x) := \int_0^T \mathrm{d}\tau \exp\left(\mathrm{i}p \cdot x(\tau) - \frac{\nu}{4} p \wedge \dot{x}(\tau) \right).$$

The integral X is evaluated as follows. First, decompose the $x(\tau)$ field into normal modes:

$$x^\mu(\tau) = x_0^\mu + \sum_{n=1}^{\infty} x_n^\mu \sin\left(\frac{n\pi\tau}{T}\right).$$

The integral over the zero-mode x_0 enforces total energy-momentum conservation. The rest yields

$$\begin{aligned} X = \int_{-\infty}^{\infty} \prod_{n=1}^{\infty} \mathrm{d}x_n \exp\Bigg[& -\frac{\pi^2}{8T} n^2 x_n^2 + \mathrm{i}\sum_{\ell=1}^{N} p_\ell x_n \sin\left(\frac{n\pi\tau_\ell}{T}\right) \\ & - \frac{1}{4}\sum_{\ell=1}^{N} p_\ell \wedge x_n \nu_\ell \frac{n\pi}{T} \cos\left(\frac{n\pi\tau_\ell}{T}\right) \Bigg]. \end{aligned}$$

The x_n integrations are of Gaussian type. Completing the exponent into squares and fixing the normalization as in the commutative case, we obtain

$$\begin{aligned} X = \left(\frac{1}{4\pi T}\right)^{d/2} \prod_{n=1}^{\infty} \exp\Bigg[& \frac{2T}{n^2\pi^2} \Bigg(\mathrm{i}\sum_{\ell=1}^{N} p_\ell \sin\left(\frac{n\pi\tau_\ell}{T}\right) \\ & + \frac{1}{4}\sum_{\ell=1}^{N} \theta \cdot p_\ell \nu_\ell \frac{n\pi}{T} \cos\left(\frac{n\pi\tau_\ell}{T}\right) \Bigg)^2 \Bigg]. \end{aligned}$$

Applying the identities

$$\begin{aligned} &\sin\left(\frac{n\pi\tau_i}{T}\right) \sin\left(\frac{n\pi\tau_j}{T}\right) \\ &\quad = \frac{1}{2}\left(\cos\frac{n\pi(\tau_i - \tau_j)}{T} - \cos\frac{n\pi(\tau_i + \tau_j)}{T} \right), \quad \text{etc.} \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{1}{4}(|x| - \pi)^2 - \frac{\pi^2}{12},$$

$$\sum_{n=1}^{\infty} \cos n(x - a) = \pi\delta(x - a) - \frac{1}{2},$$

we obtain

$$X = \left(\frac{1}{4}\pi T\right)^{d/2} \exp\left[-\frac{T}{4}\sum_{i,j=1}^{N} p_i \cdot p_j \times \left\{\left(1-\frac{|\tau_i-\tau_j|}{T}\right)^2 - \left(1-\frac{\tau_i+\tau_j}{T}\right)^2\right\} + \frac{\mathrm{i}}{8}\sum_{i,j=1}^{N} p_i \wedge p_j \nu_j T \frac{\partial}{\partial \tau_j}\left\{\left(1-\frac{|\tau_i-\tau_j|}{T}\right)^2 - \left(1-\frac{\tau_i+\tau_j}{T}\right)^2\right\} - \frac{1}{4T}\sum_{i,j=1}^{N} p_i \circ p_j \varepsilon_i \varepsilon_j\right], \quad (2.11)$$

where $\delta(\tau_i \pm \tau_j) = 0$ is used again. The differentiation with respect to τ_j in the second line of (2.11) produces both the Filk phase factor and the terms linear in the τ, which will be shown to yield precisely the generalized $\star$-products. Making use of the identities derived from the energy-momentum conservation,

$$\sum_{i,j=1}^{N} p_i \wedge p_j \nu_j \tau_i = \sum_{i,j=1}^{N} p_i \wedge p_j \varepsilon_{ij}(\tau_i+\tau_j),$$
$$\sum_{i,j=1}^{N} p_i \circ p_j \varepsilon_i \varepsilon_j = -\frac{1}{2}\sum_{i,j=1}^{N} \varepsilon_{ij}^2 p_i \circ p_j,$$

we were able to arrange the X-integral as

$$X = \left(\frac{1}{4\pi T}\right)^{d/2} \exp\left[\frac{\mathrm{i}}{2}\sum_{i<j}^{N} p_i \wedge p_j \nu_{ij} \varepsilon(\tau_{ij})\right] \times \exp\left[\frac{1}{2}\sum_{i,j=1}^{N} p_i \cdot p_j \mathcal{G}_B(\tau_i,\tau_j) - \frac{\mathrm{i}}{2T}\sum_{i,j=1}^{N} p_i \wedge p_j \varepsilon_{ij}(\tau_i+\tau_j) + \frac{1}{8T}\sum_{i,j=1}^{N} \varepsilon_{ij}^2 p_i \circ p_j\right].$$

Putting the above result into (2.10), we finally obtain the aforementioned expression, (2.8), for the N-point Green functions at one-loop order.

The N-point Green functions, (2.8), can also be obtained from rearranging the standard Feynman diagrammatics. This is elaborated in Appendix A. In comparing the two results, one should exercise caution that, upon reversing the orientation of the underlying string worldsheet, one also flips the overall sign of the phase factor in (2.8). In fact, the overall sign choice is fixed only after the orientation convention is chosen.

3 Comparison with string worldsheet computation

Having found the N-point Green functions in the worldline formalism, we now compare (2.8) and (2.9) with those

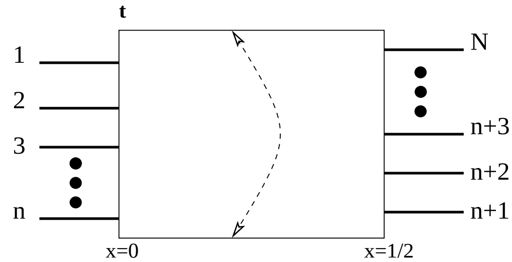

Fig. 3. Annulus as one-loop string worldsheet diagram. Tachyon vertex operators are inserted on either boundary of the annulus. In the Seiberg–Witten limit, the annulus size modulus t is scaled to infinity

obtained from the string theory computation [13]. At one loop, the relevant string worldsheet diagram is an annulus with two boundaries. We will parameterize the worldsheet by a complex coordinate, $z = x + \mathrm{i}y$, where y is periodically identified as $y \simeq y + t$ and the two boundaries are located at $x = 0$ ($\varepsilon = 1$ and $\nu = -1$, the inner boundary) and $x = 1/2$ ($\varepsilon = 0$ and $\nu = 1$, the outer boundary), respectively. Here t is the annulus modulus. External open strings can be inserted along either of the two boundaries, a direct counterpart of the twisted and untwisted interaction insertions in one-loop Feynman diagrammatics.

As we want to extract information concerning non-commutative *scalar* field theories, the relevant external string states are those of tachyons, whose vertex operator is given by

$$V_T(p,y) = g_s\sqrt{\alpha'}\mathrm{e}^{\mathrm{i}p\cdot X(y)},$$

and we turn on the constant background two-form gauge fields, which turn themselves into the non-commutativity parameter θ in the Seiberg–Witten limit. The relevant N-point tachyon S-matrix amplitude, which is depicted in Fig. 3, is schematically expressible by (up to normalization)

$$\mathcal{A} = \int_0^\infty \frac{\mathrm{d}t}{t} Z(t) \int_0^t \mathrm{d}y_1 \cdots \int_0^t \mathrm{d}y_N \times \langle V_{T1}(p_1,y_1)\cdots V_{TN}(p_N,y_N)\rangle_t = (g_s^2\alpha')^{N/2}\int_0^\infty \frac{\mathrm{d}t}{t} Z(t) \int_0^t \mathrm{d}y_1 \cdots \int_0^t \mathrm{d}y_N \times \exp\left(-\alpha'\sum_{i<j}^{N} p_i \mathcal{G}^{ij} p_j\right),$$

in terms of the worldsheet partition function $Z(t)$ and the worldsheet Green function $\mathcal{G}^{ij}$. In the case of the annulus partition function, the non-zero two-form B_{mn} background does not change the worldsheet Green function, except that the metric is replaced, in the Seiberg–Witten limit, by the open string metric G_{ab}:

$$Z(t) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\sum_{\{\mathrm{I}\}} \mathrm{e}^{-2\pi\alpha' t(k\cdot G\cdot k + M_{\mathrm{I}}^2)} = \left(\frac{1}{2\pi\alpha' t}\right)^{d/2} f_1(q)^{-24},$$

where the summation {I} is over the entire string states in the intermediate channel, and

$$f_1(q) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m), \quad \text{where} \quad q = \mathrm{e}^{-2\pi t},$$

as are relevant for the bosonic $\mathrm{D}_{(d-1)}$-branes.

The boundary worldsheet propagator has the following form [14]: for two points on the same boundary, relevant for the planar diagram contributions,

$$\mathcal{G}^{ab}_{\mathrm{p}}(z_i, z_j) = \alpha' G^{ab} \mathcal{G}(z_i, z_j) + \frac{\mathrm{i}}{2} \theta^{ab} \varepsilon(z_i - z_j), \qquad (3.1)$$

while for two points on different boundaries, relevant for the non-planar diagram contributions,

$$\mathcal{G}^{ab}_{\mathrm{np}}(z_i, z_j) = \alpha' G^{ab} \mathcal{G}(z_i, z_j) + \frac{(\theta \cdot G \cdot \theta)^{ab}}{2\pi\alpha' t} (x_i - x_j)^2 - \frac{2\mathrm{i}}{t} \theta^{ab} (x_i - x_j)(y_i + y_j). \qquad (3.2)$$

Here, the function $\mathcal{G}(z_i, z_j)$ is defined by

$$\mathcal{G}(z_i, z_j) = -\log \left| \frac{\vartheta_1(z_i - z_j | \mathrm{i}t)}{\vartheta_1'(0|\mathrm{i}t)} \right|^2 + \frac{2\pi}{t} (y_i - y_j)^2,$$

in terms of the theta function ϑ_1.

To extract the non-commutative scalar field theory amplitudes from the open string tachyon S-matrix amplitudes, we will take the Seiberg–Witten decoupling limit $\alpha' \to 0$ under which the massive string modes decouple, while the open string metric G^{ab} and non-commutativity θ^{ab} are held fixed. In fact, technically speaking, what we really do here is to isolate the loop contribution from the tachyon intermediate state. This contribution is exponentially divergent and dominates contributions from higher mass intermediate states. We then analytically continue the mass parameter m^2 to a proper positive value to match our cubic field theory mass parameter. In this process, we also keep the field theory moduli parameters T and τ fixed by putting

$$2\pi\alpha' t = T \quad \text{and} \quad 2\pi\alpha' y = \tau,$$

viz. the annulus becomes infinitely thin, making essentially a circle, relevant for a one-loop Feynman diagram. Through this procedure, one finds that the partition function $Z(t)$ turns into

$$Z(t) \to \left(\frac{1}{T}\right)^{d/2} \mathrm{e}^{-m^2 T},$$

matching the corresponding factor in the field theory result, (2.8).

The two-point function $\mathcal{G}$ in the decoupling limit is reduced to (see for instance [15])

$$-\alpha' \mathcal{G}(z_i, z_j) \longrightarrow \mathcal{G}_B = |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T},$$

viz. only the zero-mode part of ϑ_1 remains. Also, noting that $(x_i - x_j) = -\varepsilon_{ij}/2$ vanishes when the ith and jth insertions are on the same boundary, the last two terms in the non-planar propagator (3.2) are reduced to the last two terms in (2.9). Likewise, the second term in the planar propagator (3.1) gives rise to the Filk phase factors, as, when the ith and jth insertions are along the same boundary, $\varepsilon(z_i - z_j) = \varepsilon(\tau_{ij})$ at $x = 0$ ($\nu = -1$) and $\varepsilon(z_i - z_j) = -\varepsilon(\tau_{ij})$ at $x = 1/2$ ($\nu = 1$). Putting these observations together, we conclude that (2.8) and (2.9) follow precisely from the string theory computation in the Seiberg–Witten limit.

The expression (2.8) is the general one for a given value of N, the total number of external scalar insertions; the sum over $\{\nu_i\}$ is over 2^N possible terms, spanning the cases of inner or outer boundary insertion for each interaction vertex. Decomposing $N = N_1 + N_2$ where N_1 is the number of inner boundary insertions and N_2 is the number of outer boundary insertions, the terms of (2.8) can be classified into planar and non-planar contributions, depending on the value of N_1: two terms, $N_1 = 0$ or N, are planar diagrams, while $0 < N_1 < N$ are non-planar diagrams (consisting of $N!/(N_1!N_2!)$ symmetrization of the external momenta). The non-planar diagrams correspond to the double trace terms

$$\mathrm{Tr}\, \underbrace{\Phi(p_1) \cdots \Phi(p_{N_1})}_{N_1} \, \mathrm{Tr}\, \underbrace{\Phi(p_{N_1+1}) \cdots \Phi(p_N)}_{N_2}.$$

For fixed N_1, let us denote the net momentum flow between the inner and the outer boundaries

$$P = \sum_{i=1}^{N} \varepsilon_i p_i = \sum_{r=1}^{N_1} p_r. \qquad (3.3)$$

From here on, the indices $r, s, \cdots$ run from 1 to N_1 (inner insertions) while the indices $m, n, \cdots$ run from 0 to N_2 (outer insertions). Using the overall momentum conservation, we find that the contribution to the amplitudes from the third term of (2.9) can be written as

$$\exp\left(-\frac{P \circ P}{4T}\right).$$

Another useful identity that can also be proved using momentum conservation is

$$\frac{1}{2} \sum_{i,j=1}^{N} p_i \wedge p_j \varepsilon_{ij} (\tau_i + \tau_j) = \frac{1}{2} \sum_{i<j}^{N} p_i \wedge p_j (\nu_i + \nu_j) \tau_{ij}.$$

The quantity $(\nu_i + \nu_j)/2$ equals $+1$ when i and j are both outer insertions, -1 when they are both inner insertions, and 0 otherwise. Thus, for fixed N_1, each term in (2.8) can be expressed by

$$\Gamma_{N,\{\nu_j\}} = \frac{\hbar}{2} \left(-\frac{\lambda}{2}\right)^N \int_0^\infty \frac{\mathrm{d}T}{T} \left(\frac{1}{4\pi T}\right)^{d/2} \times T^{N_1+N_2} \exp\left[-m^2 T - \frac{P \circ P}{4T}\right]$$

$$\times \left(\prod_{r=1}^{N_1} \int_0^1 \mathrm{d}\tau_r \right)$$
$$\times \exp\left(-\frac{\mathrm{i}}{2} \sum_{r<s} p_r \wedge p_s \varepsilon(\tau_{rs}) + \mathrm{i} p_r \wedge p_s \tau_{rs} \right)$$
$$\times \left(\prod_{a=1}^{N_2} \int_0^1 \mathrm{d}\tau_a \right) \tag{3.4}$$
$$\times \exp\left(+\frac{\mathrm{i}}{2} \sum_{a<b} p_a \wedge p_b \varepsilon(\tau_{ab}) - \mathrm{i} p_a \wedge p_b \tau_{ab} \right)$$
$$\times \exp\left(T \sum_{i<j} p_{ia} G^{ab} p_{jb} \left(|\tau_i - \tau_j| - (\tau_i - \tau_j)^2 \right) \right),$$

where we have rescaled the τ by T so that they take values in the interval $[0, 1]$. The amplitude expression (3.4) is essentially identical to the four point amplitudes ($N_1 + N_2 = 4$) obtained in [16] in the case of the $\mathcal{N} = 4$ non-commutative $U(1)$ gauge theory up to a number of details. First, the polarization dependence of the gauge fields is deleted in the scalar field theory case. Secondly, the $(-1)^{N_1}$ factor is absent reflecting the difference in the parity under $\tau \to -\tau$ between the tachyon vertex operator (with even worldsheet oscillation number) and the gauge vertex operator (with odd worldsheet oscillation number). Third, while we had to rely on the analytic continuation to make the m value an appropriate number for the scalar theory, one can rely on the Higgs mechanism, i.e., the separation r between two parallel D-branes, to produce the mass $m = r/(2\pi\alpha')$ for the gauge theories. One further notes that the summation over $N!/(N_1!N_2!)$ terms fully symmetrizes the external momenta for each non-planar amplitude, in line with the symmetric trace prescription in non-abelian Born–Infeld theory.

To get further insight into the amplitude (3.4), we now expand the last line in (3.4) in the low-energy limit:

$$p_{ia} G^{ab} p_{jb} \ll m^2 \quad \text{for every } i, j.$$

In this limit, being subdominant compared to the first line, the last line in (3.4) simply drops out. The leading term in this expansion exhibits the factorization property [17] manifestly:

$$\Gamma_{N,\{\nu_j\}} = \frac{\hbar}{2} \left(-\frac{\lambda}{2} \right)^{\mathrm{N}} \int_0^\infty \frac{\mathrm{d}T}{T} \left(\frac{1}{4\pi T} \right)^{d/2}$$
$$\times T^{N_1+N_2} \exp\left[-m^2 T - \frac{P \circ P}{4T} \right]$$
$$\times \left(\prod_{r=1}^{N_1} \int_0^1 \mathrm{d}\tau_r \right)$$
$$\times \exp\left(-\frac{\mathrm{i}}{2} \sum_{r<s} p_r \wedge p_s \varepsilon(\tau_{rs}) + \mathrm{i} p_r \wedge p_s \tau_{rs} \right)$$
$$\times \left(\prod_{a=1}^{N_2} \int_0^1 \mathrm{d}\tau_a \right) \tag{3.5}$$
$$\times \exp\left(+\frac{\mathrm{i}}{2} \sum_{a<b} p_a \wedge p_b \varepsilon(\tau_{ab}) - \mathrm{i} p_a \wedge p_b \tau_{ab} \right).$$

The effective action is then obtained by computing the moduli parameter integrals explicitly and then summing over $\Gamma_{N,\{\nu_j\}}$ along with the combinatorial weight $1/N!$, as explained above (2.7). We elaborate the details in the next section.

4 Effective action, $\star_n$-products and open Wilson lines

Now we begin with the evaluation of the moduli parameter integrals of the factorized low-energy expression, (3.5). As elaborated in the previous section, by rescaling the vertex position moduli τ's by $T \cdot \tau$, the moduli integrals in T, τ_r, τ_a are also factorized. Thus, we evaluate first the T-modulus integral. Recall that the T-modulus corresponds, in the open string worldsheet computation, to the modulus of an annulus diagram. One readily obtains

$$\begin{aligned} \mathcal{K}_N(P, \Lambda; d) &:= 2^N \int_0^\infty \frac{\mathrm{d}T}{T} \left(\frac{1}{4}\pi \mathrm{T} \right)^{d/2} \\ &\times\ T^N \exp\left[-m^2 T - \frac{P \circ P + \Lambda^{-2}}{4T} \right] \\ &= 2 \left(\frac{1}{2\pi} \right)^{d/2} \left(\frac{P \circ P + \Lambda^{-2}}{m^2} \right)^{(N/2)-(d/4)} \\ &\times\ \mathcal{K}_{N-(d/2)} \left(m |P \circ P + \Lambda^{-2}|^{1/2} \right), \end{aligned} \tag{4.1}$$

where we have introduced the UV cutoff Λ explicitly, and the dependence on P, Λ, and the spacetime dimension d are emphasized. The function $\mathcal{K}_{-(d/2)+N}(z)$ refers to the modified Bessel function. For the planar diagrams, inferred from (3.3), $P = 0$ and hence the UV cutoff Λ is indispensable.

Next, evaluate the moduli integrals in the second and the third lines in (3.5). These integrals turn out to be identical to the definition of generalized $\star_N$-products, as defined, for instance, in [12]. Note that we have decomposed the N-point interaction vertices into N_1 untwisted ones and N_2 twisted ones, where $N = N_1 + N_2$. In the string worldsheet computation, the former type of insertions corresponds to the "outer" boundary insertions, and the latter to the "inner" boundary insertions. One readily finds that each group of the insertions yields a cluster of generalized $\star$-products. The T-integral, $\mathcal{K}_N(P, \Lambda; d)$, then supplies a sort of "propagator", connecting the two clusters of generalized $\star$-products; see Fig. 1.

As emphasized already, the generalized $\star$-product arises when there exists a net momentum flow between the two clusters of external lines, viz. between untwisted and twisted interaction vertices. As denoted in (3.3), the net momentum flow P is given by

$$P + p_1 + \cdots + p_{N_2} = 0.$$

Making use of the identity

$$\sum_{a<b=1}^{N_2} p_a \wedge p_b(\tau_a - \tau_b) = \sum_{a=1}^{N_2} P \wedge p_a \tau_a,$$

we can re-express the third line of (3.5) by

$$\begin{aligned}
&\left(\prod_{a=1}^{N_2}\int_0^1 \mathrm{d}\tau_a\right) \exp \sum_{a<b=1}^{N_2} \left(\frac{\mathrm{i}}{2}\varepsilon(\tau_{ab}) p_a \wedge p_b - \mathrm{i}\tau_{ab} p_a \wedge p_b\right) \\
&= \left(\prod_{a=1}^{N_2}\int_0^1 \mathrm{d}\tau_a\right) \exp \sum_{a<b=1}^{N_2} \left(\frac{\mathrm{i}}{2}\varepsilon(\tau_{ab}) p_a \wedge p_b\right) \\
&\times \exp\left(-\mathrm{i}\sum_a P \wedge p_a \tau_a\right). \qquad (4.2)
\end{aligned}$$

As expressed, the moduli integrals over the τ are unordered and range over the entire circle $[0,1]$. One can decompose these integrals into $N_2!$ ordered integrals, each of which is defined with a definite ordering among the N_2 τ_a moduli parameters. For each ordering, the first exponential in (4.2) gives rise to Filk's phase factor, which, in the absence of the second exponential, simply yields the standard $\star$-product. In the case of the planar contribution, $P = 0$ and the relevant product is the *symmetrized* form of the standard $\star$-product:

$$[A_1 A_2 \cdots A_N]_{\star_{\mathrm{sym}}} := \frac{1}{N!}\sum_{\{\mathrm{perm}\}} A_{i1} \star \cdots \star A_{iN},$$

where the summation is over $N!$ permutations. In the case of the non-planar contributions, however, because of the non-vanishing momentum flow P, the relevant product turns out to be the generalized $\star_N$-product[2].

The explicit evaluation of (4.2), including the non-abelian Chan–Paton factor, was made in [17]. The results are

$$\begin{aligned}
&\mathrm{Tr}\,[f_1(p_1), f_2(p_2), \cdots f_{N_2}(p_{N_2})]_{\star_{N_2}} \\
&= \sum_{(\mathrm{N}_2-1)!} f_1^{a_1}(p_1)\cdots f_{N_2}^{a_{N_2}}(p_{N_2})\mathrm{Tr}\,(\mathrm{T}^{a_1}\cdots T^{a_{N_2}}) \\
&\times \left(\frac{\exp\left[\frac{\mathrm{i}}{2}\sum_{i<j}^{N_2} p_i \wedge p_j\right]}{\prod_{i=2}^{N_2}(-\mathrm{i}k \wedge P_i)} + (\text{cyclic permutations}) \right),
\end{aligned}$$

where the summation runs over the $(N_2-1)!$ non-cyclic permutations (with independent Chan–Paton factor), $f_i = \sum_{a_i} f_i^{a_i} T^{a_i}$, T^{a_i} are generators of the $U(n)$ Chan–Paton group, and $P_i := \sum_{j=i}^{N} p_j$. For $U(1)$ gauge group, they reduce to

$$\begin{aligned}
&[f_1(p_1), f_2(p_2), \cdots f_{N_2}(p_{N_2})]_{\star_{N_2}} \\
&= \sum_{(\mathrm{N}_2-1)!} f_1(p_1)\cdots f_{N_2}(p_{N_2}) \\
&\times \left(\frac{\exp\left[\frac{\mathrm{i}}{2}\sum_{i<j}^{N_2} p_i \wedge p_j\right]}{\prod_{i=2}^{N_2}(-\mathrm{i}k \wedge P_i)} + (\text{cyclic permutations}) \right).
\end{aligned}$$

One can explicitly work all these out and find that they are given by

$$\begin{aligned}
&[A(x_1)B(x_2)]_{\star_2} := \frac{\sin\left(\frac{1}{2}\partial_1 \wedge \partial_2\right)}{\frac{1}{2}\partial_1 \wedge \partial_2} A(x_1)B(x_2), \\
&[A(x_1)B(x_2)C(x_3)]_{\star_3} \qquad (4.3) \\
&:= \left[\frac{\sin\left(\frac{1}{2}\partial_2 \wedge \partial_3\right)}{\frac{1}{2}(\partial_1+\partial_2)\wedge\partial_3} \frac{\sin\left(\frac{1}{2}\partial_1 \wedge (\partial_2+\partial_3)\right)}{\frac{1}{2}\partial_1 \wedge (\partial_2+\partial_3)} + (1 \leftrightarrow 2) \right] \\
&\times A(x_1)B(x_2)C(x_3),
\end{aligned}$$

and so forth. Evidently, as the k subset of the momenta go to zero, $\star_N$ is reduced to $\star_{N-k}$.

Combining (4.1) and (4.3), (3.5) can be rewritten

$$\begin{aligned}
\Gamma_{N,\{\nu_j\}} &= \frac{\hbar}{2}\left(-\frac{\lambda}{4}\right)^{N_1+N_2} [\Phi\cdots\Phi]_{\star_{N_1}}(-P) \\
&\times \widetilde{\mathcal{K}}_{N-d/2}\left(P \circ P + 1/\Lambda^2\right) [\Phi\cdots\Phi]_{\star_{N_2}}(+P), \quad (4.4)
\end{aligned}$$

where the kernel $\mathcal{K}_n$ is given, from (4.1), by

$$\tilde{\mathcal{K}}_n(z^2) = 2\left(\frac{1}{2\pi}\right)^{d/2}\left(\frac{|z|}{m}\right)^n \mathcal{K}_n(m|z|).$$

In (4.4), we have retained the UV cutoff Λ, as the result is equally valid for planar contributions in so far as P is set to zero and the generalized $\star$-products are replaced by the standard $\star$-products.

As is clear from the defintion in (4.3), the generalized star products are symmetric with respect to the external momenta. Hence, each summand in the $2^{N_1+N_2}$-summations over $\{\nu_j\}$ yield the same contribution as long as N_1 (thus N_2) is the same; this gives rise to the combinatorial factor $N!/(N_1!N_2!)$. Recalling the definition of the effective action in our convention with the $1/N!$ factor, we finally obtain the one-loop effective action:

$$\Gamma[\Phi] = \frac{\hbar}{2}\sum_{N=1}^{\infty}\left(-\frac{\lambda}{4}\right)^N \frac{1}{N!}\int \mathrm{d}^d x \sum_{N_1=0}^{N} \frac{N!}{N_1!(N-N_1)!}$$

[2] The relevance of generalized $\star$-products and the relation to gauge invariance and the Seiberg–Witten map have recently been studied [18], but all in the context of non-commutative *gauge* theories

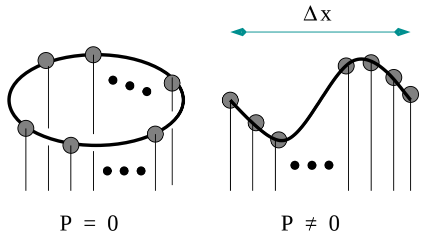

Fig. 4. Spacetime snapshot of planar and non-planar contributions. The partial momentum sum of the (un)twisted interaction vertices is denoted P. For $P = 0$, viz. the planar contribution, the virtual quanta sweep a closed path in spacetime. When $P \neq 0$, viz. the non-planar contribution, the virtual quanta jump Δx in spacetime

$$\times \left[\Phi\cdots\Phi\right]_{\star_{N_1}}(x)\widetilde{\mathcal{K}}_{N-d/2}(-\partial_x\circ\partial_x+\Lambda^{-2})$$
$$\times \left[\Phi\cdots\Phi\right]_{\star_{N-N_1}}(x),$$

encompassing both planar ($N_1 = 0$ or N) and non-planar contributions. As we will see below, the planar contribution is the counterpart of the closed Wilson loop one-point function, viz. a counterpart of the closed string tadpole, while the non-planar contribution is the counterpart of the open Wilson line two-point functions; see Fig. 4.

4.1 Planar contribution

We first consider the planar contribution $N_1 = 0$ or N in the limit where the momentum cutoff is much larger than the mass scale m, $\Lambda \gg m$. From the Taylor expansion of the modified Bessel function, we obtain the following expressions for the kernel:

$$\mathcal{K}_n(z^2) = 2\left(\frac{1}{2\pi}\right)^{d/2}\left(\frac{|z|}{m}\right)^n \frac{\Gamma(|n|)}{2}\left(\frac{2}{m|z|}\right)^{|n|}$$

for $n \neq 0$ and

$$\mathcal{K}_0(z^2) = 2\left(\frac{1}{2\pi}\right)^{d/2}\left(-\log\frac{m|z|}{2}\right)$$

for $n = 0$ and $n = N - d/2$. Here, $z = 1/\Lambda$, and, as explained above, the generalized star products should be understood as the standard star products:

$$\left[\Phi,\cdots,\Phi\right]_{\star_{N_1}}(x)\left[\Phi,\cdots,\Phi\right]_{\star_{N-N_1}}(x) = \left[\Phi\star\cdots\star\Phi\right](x),$$

viz. an N-tuple of standard $\star$-products, etc. Details of the UV divergence depend on the spacetime dimension d. For instance, in $d = 6$, where the theory is renormalizable, the two-point Green function ($N = 2$) is quadratically divergent, and the three-point Green function ($N = 3$) is logarithmically divergent. The higher-point functions ($n > 0$) are finite, as the dependence on the UV cutoff Λ cancels out. One furthermore observes that, after renormalization of the divergent contributions, the combinatorial factors turn out to be

$$\frac{(\mathrm{N}-4)!}{N!} = \frac{1}{N(N-1)(N-2)(N-3)}$$
$$= -\frac{1}{6}\left(\frac{1}{N}-\frac{3}{N-1}+\frac{3}{N-2}-\frac{1}{N-3}\right).$$

Thus, the planar contribution to the effective action is given by

$$\Gamma_{\mathrm{p}} \simeq \hbar\left(m^2+\frac{\lambda}{2}\Phi\right)^3_\star \star\log_\star\left(m^2+\frac{\lambda}{2}\Phi\right). \tag{4.5}$$

The result (4.5) is precisely the non-commutative version of the "Coleman–Weinberg"-type potential, where the parameters and the fields are to be understood as renormalized ones.

The planar contribution ought to correspond, in string theory, to the diagrams with a tadpole insertion [19]. This is evidently so, except for one puzzling point: in the large non-commutativity limit, the Weyl–Moyal correspondence permits one to map the non-commutative $\lambda[\Phi^3]_\star$-theory into the large-N limit of the $U(N)$ matrix $\lambda\mathrm{Tr}[\Phi]^3$-theory. One would have expected that the dominant dynamics is describable in terms of the standard Wilson loop operators

$$W_0[\Phi] := \int \mathrm{d}^4x\,\exp_\star\left(-\lambda\Phi(x)\right),$$

where the integration is over the non-commutative directions. Typically, these Wilson loops are the large-N limit "master" fields in matrix-valued field theories. Apparently, the result, (4.5), does not involve the above Wilson loops, even after taking the large non-commutativity limit. Whether this discrepancy invalidates the concept of the master field in this context or not is unclear yet.

4.2 Non-planar contribution

The behavior of the non-planar contribution is markedly different from those of the planar part, especially as we take the large non-commutativity limit. From here on, we will drop the cutoff by sending it to infinity and Wick rotate back to the Minkowski space. The non-planar part of the effective action then becomes a double sum involving generalized $\star$-products:

$$\Gamma_{\mathrm{np}} = \frac{\hbar}{2}\sum_{N=2}^{\infty}\left(-\frac{\lambda}{4}\right)^N\frac{1}{N!}\int\mathrm{d}^dx\sum_{n=1}^{N-1}\binom{N}{n}\left[\Phi\cdots\Phi\right]_{\star_n}(x)$$
$$\times\,\mathcal{K}_{N-(d/2)}\left(-\partial_x\circ\partial_x\right)\left[\Phi\cdots\Phi\right]_{\star_{N-n}}(x).$$

To proceed, we will be taking the low-energy, large non-commutativity limit:

$$q_\ell \sim \epsilon,\quad \mathrm{Pf}\theta \sim \frac{1}{\epsilon^2}\quad \text{as}\quad \epsilon\to 0, \tag{4.6}$$

so that

$$q_\ell\cdot q_m \sim \mathcal{O}(\epsilon^{+2}) \to 0,\quad q_\ell\wedge q_m \to \mathcal{O}(1),$$
$$q_\ell\circ q_m \sim \mathcal{O}(\epsilon^{-2}) \to \infty. \tag{4.7}$$

In this limit, the modified Bessel function $\mathcal{K}_n$ exhibits the following asymptotic behavior:

$$\mathcal{K}_n(mz) \to \sqrt{\frac{\pi}{2mz}}\mathrm{e}^{-m|z|}\left[1+\mathcal{O}\left(\frac{1}{m|z|}\right)\right].$$

Most remarkably, the asymptotic behavior is *independent* of the index n. Compared to the planar effective action, there is not the extra $n!$ factor that (partially) cancels $N!$ in the denominator, which shows that the summed form of the effective action markedly changes. In the low-energy limit, the Fourier-transformed kernels $\widetilde{\mathcal{K}_n}$ obey the following recursive relation:

$$\widetilde{\mathcal{K}}_{n+1}(k\circ k) = \left(\frac{|\theta\cdot k|}{m}\right)\widetilde{\mathcal{K}}_n(k\circ k),$$

viz.

$$\widetilde{\mathcal{K}_n}(k\circ k) = \left(\frac{|\theta\cdot k|}{m}\right)^n \widetilde{\mathcal{Q}}(k\circ k).$$

Here, the kernel $\widetilde{\mathcal{Q}}$ is given by

$$\widetilde{\mathcal{Q}}(k\circ k) = (2\pi)^{(1-d)/2}\left|\frac{1}{m\theta\cdot k}\right|^{1/2}\exp(-m|\theta\cdot k|).$$

Note that, in the power-series expansion of the effective action, a natural expansion parameter is $|\theta\cdot k|$.

Thus, the non-planar one-loop effective action in momentum space is expressible as

$$\begin{aligned}\mathit{\Gamma}_{\mathrm{np}}[\Phi] &= \frac{\hbar}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\widetilde{\mathcal{K}}_{-d/2}(k\circ k)\sum_{N=2}^{\infty}\sum_{n=1}^{N-1}\left(-\frac{\lambda}{4m}\right)^N \\ &\times\left(\frac{1}{n!}|\theta\cdot k|^n\left[\widetilde{\Phi}\cdots\widetilde{\Phi}\right]_{\star_n}[k]\right) \\ &\times\left(\frac{1}{(N-n)!}|\theta\cdot k|^{N-n}\left[\widetilde{\Phi}\cdots\widetilde{\Phi}\right]_{\star_{N-n}}[-k]\right). \qquad (4.8)\end{aligned}$$

Utilizing the relation between the generalized $\star_n$ products and the *scalar* open Wilson line operators, as elaborated in [1], the non-planar one-loop effective action can be summed up into a remarkably simple closed form. Denote the rescaled coupling parameter by $g := \lambda/4m$ (see (1.2)). Then, because of the algebraic relation $[\widetilde{\Phi}\star_0\widetilde{\Phi}]_k = (2\pi)^d\delta^{(d)}(k)$, the domain of the double summations can be extended to $n=0$, $N=0$ terms, as they yield an identically vanishing contribution *after* the k integration is performed. Once this arrangement is made, partial summations over N and n can be performed explicitly. Exploiting the exchange symmetry $n\leftrightarrow(N-n)$, the summation domain (n,N) over the lower triangular lattice points can be mapped to the one over the upper triangular lattice points. By averaging over the two summation domains, one can then rearrange the double summations into *decoupled* ones over n and $(N-n)$. One finally obtains

$$\mathit{\Gamma}_{\mathrm{np}}[\Phi] = \frac{\hbar}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}W_k[\Phi]\cdot\widetilde{\mathcal{K}_{-d/2}}(k\circ k)\cdot W_{-k}[\Phi],$$

yielding precisely the aforementioned result, (1.1).

5 Conclusions and discussions

In this paper, we have studied the one-loop effective action in the non-commutative $\lambda[\Phi^3]_\star$-theory. In order to make a direct comparison with the Seiberg–Witten limit of the open string worldsheet formulation, in computing the one-particle irreducible one-loop Green functions, we have adopted the worldline formulation of the theory. We have observed that, at low energy, the one-loop diagrams, both planar and non-planar, are completely factorizable. We have shown explicitly that, while the planar contribution is expressed in terms of the standard $\star$-products, the non-planar contribution is expressible solely in terms of the generalized $\star$-products. This implies that the structure of the one-loop effective action reveals quite different physics between planar and non-planar contributions. In particular, we were able to show that, in the large non-commutativity limit, the non-planar contribution is expressible in terms of *open* Wilson line operators, thus completing the proof of our earlier result in [1]. The planar contribution, on the other hand, gives rise to a non-commutative version of the Coleman–Weinberg-type potential, in contrast to the anticipation that the planar part ought to be expressible in terms of Wilson loop operators – the putative master field in the planar limit of matrix-valued field theories. The next obvious step is to extend the computation to two loops and confirm that the two-loop effective action is re-expressible in terms of (at most) three open Wilson line operators. We will report the result in a separate publication.

Our computation in the worldline formulation has shed on several new points light concerning the spacetime interpretation of the one-loop physics. Among them is the shift of the momentum integration variable, as is evident from the integration K in Sect. 2.2. It suggests that a variation of the internal momemtum Δk^a integrated along the entire loop amounts to

$$\begin{aligned}&\Delta\int_0^T \mathrm{d}\tau k^a(\tau) \\ &= \frac{\mathrm{i}}{2}\int_0^T\mathrm{d}\tau\left\{\dot{x}^a(\tau)-\frac{1}{2}\sum_{j=1}^N\theta^{ab}p_{bj}\nu_j\delta(\tau-\tau_j)\right\} \\ &= \frac{\mathrm{i}}{2}\sum_{j=1}^N\theta^{ab}p_{bj}\varepsilon_j. \qquad (5.1)\end{aligned}$$

Recalling ε_j take either the value 0 or 1, depending on whether the jth vertex is an untwisted or twisted insertion, we recognize that the above relation is precisely the momentum-space counterpart of the dipole relation (1.3). Recalling that $k^a(\tau)$ is the conjugate momentum to $x^a(\tau)$, both of which are associated with the virtual quanta circulating around the loop, the above relation asserts that the virtual quanta is not a point-like constituent, obeying the standard Fourier transformation relation between $x^a(\tau)$ and $k^a(\tau)$, but behaves as a sort of rigid rod whose size is proportional to the momentum. We trust details of this unusual physics – physics of dipoles – deserve further investigation and we intend to report on new understanding of this aspect in separate publications.

Acknowledgements. We thank H. Liu, P. Mayr, Y. Oz, and L. Susskind for discussions. SJR acknowledges warm hospitality of Henri Poincaré Institut, Institut des Hautes Etudes Scientifiques, Theory Division at CERN, and Department of Mathematics and Physics at Amsterdam University at various stages of this project.

Appendix
A Schwinger parameterization of one-loop Feynman diagram

In this section, we will provide a check point of the worldline formulation introduced in Sect. 2 with the standard Feynman diagrammatics. Although a general expression for N-point, one-loop Feynman diagrams are given, for instance, in [20,9], they are not in a convenient form for comparison with the results in the worldline formulation, mainly because of different moduli parameterizations and the omission of overall normalization and combinatorial factors (which are necessary for the resummation of the N-point Green functions into the effective action). Let us consider the N-point Feynman diagram (cf. Fig. 2), wherein the non-planar phase factors $\mathrm{e}^{\mathrm{i}k\wedge p_1}$, $\mathrm{e}^{\mathrm{i}(k+p_1)\wedge p_2}$, $\mathrm{e}^{\mathrm{i}(k+p_1+p_2)\wedge p_3}, \cdots, \mathrm{e}^{\mathrm{i}k\wedge p_N}$ are inserted at each of the N interaction vertices, respectively, as well as the overall Filk phase factor $\mathrm{e}^{-(\mathrm{i}/2)\sum_{i<j} p_i\wedge p_j}$. The one-loop Feynman amplitude is given by

$$F_N = \mathrm{e}^{-(\mathrm{i}/2)\sum_{i<j} p_i\wedge p_j} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \times \frac{\mathrm{e}^{\mathrm{i}k\wedge P}\mathrm{e}^{\mathrm{i}\sum_{l=2}^{N-1}\sum_{i=2}^{l-1} p_i\wedge p_l \varepsilon_l}}{k^2(k+p_1)^2(k+p_1+p_2)^2\cdots(k+p_1+p_2+\cdots+p_{N-1})^2}. \quad (A.1)$$

Here, the twist factor $\varepsilon_i = 1$ or 0; $i = 1, \cdots, N$ are inserted for the planar and the non-planar vertex insertions, respectively, and

$$P^a = \sum_{i=1}^N \varepsilon_i p_i^a. \quad (A.2)$$

We rewrite the momentum integral in terms of the overall modulus integral (global Schwinger parameter $T \equiv \tau_N$) and $(N-1)$ relative moduli integrals (local Schwinger parameters, τ_i):

$$F_N = \prod_{i<j=1}^N \mathrm{e}^{-(\mathrm{i}/2)p_i\wedge p_j} \prod_{k=2}^{N-1}\prod_{\ell=2}^{k-1} \mathrm{e}^{\mathrm{i}p_\ell\wedge p_k\varepsilon_k} \int_0^\infty \mathrm{d}T \left(\frac{1}{4\pi T}\right)^{d/2} \times \prod_{n=1}^{N-1}\int_0^{\tau_{n+1}} \mathrm{d}\tau_n \exp\left[\frac{1}{T}\left(\sum_{j=1}^{N-1}\tau_{jj+1}\sum_{\ell=1}^j p_\ell + \frac{\mathrm{i}}{2}\theta\cdot P\right)^2 - \sum_{j=1}^{N-1}\tau_{jj+1}\left(\sum_{\ell=1}^j p_\ell\right)^2\right], \quad (A.3)$$

where $\tau_{ij} = \tau_i - \tau_j$. From energy-momentum conservation, the following relations can be deduced:

$$P\circ P = -\sum_{i<j}^N \varepsilon_{ij}^2 p_i \circ p_j,$$

$$\sum_{j=1}^{N-1}\tau_{jj+1}\sum_{l=1}^j p_\ell \wedge P = \sum_{i<j}^N (\tau_i+\tau_j)\varepsilon_{ji} p_i\wedge p_j = -\sum_{i<j}^N \sum_{k=1,k\neq i,j}^N \tau_k \varepsilon_{ji} p_i \wedge p_j,$$

$$\frac{1}{T}\left(\sum_{j=1}^{N-1}\tau_{jj+1}\sum_{\ell=1}^j p_\ell\right)^2 - \sum_{j=1}^{N-1}\tau_{jj+1}\left(\sum_{\ell=1}^j p_\ell\right)^2 = \sum_{i<j}^N p_i\cdot \mathcal{G}_B(\tau_j,\tau_i)\cdot p_j,$$

and

$$\sum_{i<j}^N \varepsilon_{ij} p_i\wedge p_j = \sum_{i<j}^N (\nu_i-\nu_j) p_i\wedge p_j = \sum_{i<j}^N \tau_{ij} p_i \wedge p_j = 0. \quad (A.4)$$

Utilizing the identity (A.4), the non-planar phase factor can be simplified to

$$\mathrm{e}^{\mathrm{i}\sum_{l=2}^{N-1}\sum_{i=2}^{l-1} p_i\wedge p_l\varepsilon_l} = \mathrm{e}^{\mathrm{i}\sum_{i<j}\varepsilon_i p_i\wedge p_j},$$

thus deriving

$$F_N = \mathrm{e}^{-(\mathrm{i}/2)\sum_{i<j}\nu_j p_i\wedge p_j} \int_0^\infty \mathrm{d}T \left(\frac{1}{4\pi T}\right)^{d/2} \prod_{n=1}^{N-1}\int_0^{\tau_{n+1}} \mathrm{d}\tau_n \times \exp\left[\sum_{i<j}^N p_i\cdot\mathcal{G}_{B\theta}(\tau_i,\tau_j;\varepsilon_i,\varepsilon_j)\cdot p_j\right]. \quad (A.5)$$

Using the above identities again, Filk's overall phase factor in (A.5) can be reduced to

$$-\frac{\mathrm{i}}{2}\sum_{i<j} p_i\wedge p_j \nu_j = -\frac{\mathrm{i}}{2}\sum_{i<j} p_i\wedge p_j \frac{1}{2}(\nu_i+\nu_j),$$

and

$$F_N = \int_0^\infty \mathrm{d}T\left(\frac{1}{4\pi T}\right)^{d/2} \mathrm{e}^{-m^2 T}\prod_{n=1}^{N-1}\int_0^{\tau_{n+1}}\mathrm{d}\tau_n \times \prod_{i<j}^N \exp\left(\frac{\mathrm{i}}{2}p_i\wedge p_j \frac{1}{2}(\nu_i+\nu_j)\varepsilon(\tau_{ji})\right) \times \exp\left[\sum_{i<j}^N p_i\cdot\mathcal{G}_{B\theta}(\tau_i,\tau_j;\varepsilon_i,\varepsilon_j)\cdot p_j\right]. \quad (A.6)$$

Here, having in mind the generalization to all possible orderings, we have attached $\varepsilon(\tau_{ji})$ to the phase factor in order to blindly count the ordering dependences. If we shuffle the external legs, we obtain different diagrams with different phase factors. Thus all ordered integrals correspond to all possible distinct Feynman diagrams. In this way, we recover the full integral regions after summing all contributions for fixed ν_i. Obviously, we then also have to sum over all combinations of ν_i. As in Sect. 2.2 (2.8) is invariant under the simultaneous constant shifts of all τ (by noticing (2.9) and (A.4)), we can fix one of them in (2.8); for example, $\tau_N = T$, as has been assumed throughout this section, and reproduce

$$\Gamma_N = C(-\lambda)^N \sum_{\nu_j} \int \mathrm{d}T \mathrm{e}^{-m^2 T} \left(\frac{1}{4\pi T}\right)^{d/2} \times \left(\prod_{i=1}^{N-1} \int_0^T \mathrm{d}\tau_i\right) \mathrm{e}^{(\mathrm{i}/4)\sum_{i<j} p_i \wedge p_j (\nu_i+\nu_j)\varepsilon(\tau_{ij})} \times \exp\left[\frac{1}{2}\sum_{i,j=1}^N p_i \cdot \mathcal{G}_{B\theta}(\tau_i,\tau_j;\varepsilon_i,\varepsilon_j)\cdot p_j\right].$$

Here, C is the normalization factor defined by the fraction of the symmetry factor S_N and the number of topologically distinct integration regions C_N:

$$C = \frac{S_N}{C_N}. \tag{A.7}$$

Following closely the arguments of [21], we will now determine the combinatorial factor C_N. If one expands the effective action in powers of the coupling constant, the combinatorial weight for the expansion of the Feynman diagrams is obtained by shuffling the external interaction vertices. One obtains

$$w = \frac{S_N n^T}{N!}, \tag{A.8}$$

where n^T is the number of the topologically distinct diagrams out of these shuffled diagrams; S is the symmetry factor for them. In the present case, this number w is given by

$$w = \frac{1}{2\cdot N\cdot 2^N}. \tag{A.9}$$

It comprises the trace-log factor 1/2, the $1/N$ coming from the Taylor expansion of the one-loop form $\ln(1+x)$, and the coupling factor 2^{-N}. (One may simply multiply $2^{-\mathrm{N}}$ with the result (3.25) of [21]). Combining (A.8) and (A.9), we obtain

$$S_N = \frac{(N-1)!}{2^{N+1} n^T}.$$

As C_N is defined by

$$C_N = \frac{(N-1)!}{n^T},$$

we can put these into (A.7) and obtain

$$C = \frac{1}{2\cdot 2^N},$$

therefore proving that our worldline master formula coincides with the Feynman diagrammatics result.

The commutative limit, $\theta^{ab} \to 0$, also comes out correct, as all the 2^N terms of the $\{\nu_i\}$ summation are reduced to the same, single contribution, reproducing the commutative result.

B Worldline formulation of $\lambda[\Phi]^3$-theory: Review

In this appendix, we recapitulate the worldline formulation of commutative $\lambda[\Phi]^3$-theory, summarizing aspects relevant for extension to a non-commutative setup. In the background field method, the classical action is expanded to

$$S_{\mathrm{C}} = \int \mathrm{d}^d x \frac{1}{2}\varphi(x)\{-\partial^2 + m^2 + \lambda\Phi_0(x)\}\varphi(x) + \mathcal{O}(\varphi^3).$$

The one-loop effective action can be expressed, in a path integral representation, by

$$\Gamma = \frac{\hbar}{2}\int_0^\infty \frac{\mathrm{d}T}{T} \int_{x(T)=x(0)} \mathcal{D}x(\tau) \int_{k(T)=k(0)} \mathcal{D}k(\tau) \times \mathcal{P}_\tau \exp\left(-\int_0^T \left[k^2 + m^2 - \mathrm{i}k\cdot\dot{x} + \lambda\Phi_0(x(\tau))\right]\mathrm{d}\tau\right).$$

Instead of taking the obvious route of integrating over $k(\tau)$, which is a Gaussian integral, bearing in mind the application to a non-commutative setup, we will leave it as is and expand the background field Φ_0 first:

$$\Gamma = \frac{\hbar}{2}\int_0^\infty \frac{\mathrm{d}T}{T} \int_{x(T)=x(0)} \mathcal{D}x(\tau) \int_{k(T)=k(0)} \mathcal{D}k(\tau) \times \exp\left[-\int_0^T (k^2 + m^2 - \mathrm{i}k\cdot\dot{x})\mathrm{d}\tau\right] \times \sum_{N=0}^\infty (-\lambda)^N \prod_{\ell=1}^N \int_0^{\tau_{l+1}} \mathrm{d}\tau_\ell \Phi_0(x(\tau_\ell)), \quad \text{with} \quad \tau_{N+1} = T.$$

The Feynman diagrams relevant for N-point Green functions are generated automatically once the external field is replaced by "plane waves" as follows:

$$\Phi_0(x(\tau)) \to \sum_{\ell=1}^N \exp[\mathrm{i}p_\ell \cdot x(\tau)].$$

Specifically, the substitution generates all possible orderings of the moduli of interaction vertices, τ_ℓ, along the worldline one loop, discarding double insertions of the same momenta. It also generates, however, an extra $N!$ factor, as Φ_0 ought to represent a single insertion of the interaction vertex. One thus assigns $1/N!$ at the Nth order term in the effective action. These ordered τ integrals are then summable over the full integration regions:

$$\Gamma[p_1,\cdots,p_N] = \frac{\hbar}{2}\int_0^\infty \frac{\mathrm{d}T}{T} \int\limits_{x(T)=x(0)} \mathcal{D}x(\tau) \int\limits_{k(T)=k(0)} \mathcal{D}k(\tau) \times \exp\left(-\int\limits_0^T (k^2+m^2-\mathrm{i}k\cdot\dot{x})\mathrm{d}\tau\right) \times \sum_{N=0}^\infty \frac{(-\lambda)^N}{N!}\prod_{\ell=1}^N \int\limits_0^T \mathrm{d}\tau_\ell \mathrm{e}^{\mathrm{i}p_\ell\cdot x(\tau_\ell)}.$$

The N-point Green functions, $\Gamma^{(N)}$, are defined by

$$\Gamma_{1-\mathrm{loop}} = \hbar \sum_{N=0}^\infty \frac{1}{N!}\Gamma^{(N)}. \tag{B.1}$$

Integrating over $k(\tau)$ first yields

$$\Gamma^{(N)} = \frac{1}{2}(-\lambda)^N \int_0^\infty \frac{\mathrm{d}T}{T}\mathcal{N}(T)\mathrm{e}^{-m^2T} \times \int\limits_{x(T)=x(0)} \mathcal{D}x(\tau)\mathrm{e}^{-(1/4)\int_0^T \dot{x}^2\mathrm{d}\tau}\prod_{\ell=1}^N \int\limits_0^T \mathrm{d}\tau_\ell \mathrm{e}^{\mathrm{i}p_\ell\cdot x(\tau_\ell)},$$

where the normalization factor $\mathcal{N}(T)$ is defined by

$$\mathcal{N}(T) = \int \mathcal{D}k\mathrm{e}^{-\int_0^T k^2\mathrm{d}\tau}$$

and

$$\mathcal{N}(T)\int\limits_{x(0)=x(T)} \mathcal{D}x\mathrm{e}^{-(1/4)\int_0^T \dot{x}^2\mathrm{d}\tau} = \left(\frac{1}{4\pi T}\right)^{d/2}.$$

Integrating over $x(\tau)$, one obtains

$$\Gamma^{(N)} = \frac{1}{2}(-\lambda)^N \int \frac{\mathrm{d}T}{T}\left(\frac{1}{4\pi T}\right)^{d/2}\mathrm{e}^{-m^2T}\prod_{l=1}^N \int\limits_0^T \mathrm{d}\tau_\ell \times \exp\left[\sum_{i<j}^N p_i\cdot\mathcal{G}_B(\tau_i,\tau_j)\cdot p_j\right], \tag{B.2}$$

where $\mathcal{G}_B$ is the one-loop bosonic worldline propagator:

$$\mathcal{G}_B^{ab}(\tau_i,\tau_j) = G^{ab}\left[|\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T}\right].$$

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